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Higher-order Bernoulli, Euler and Hermite polynomials

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Abstract

In (Kim and Kim in J. Inequal. Appl. 2013:111, 2013; Kim and Kim in Integral Transforms Spec. Funct., 2013, doi:10.1080/10652469.2012.754756), we have investigated some properties of higher-order Bernoulli and Euler polynomial bases in $\mathbb{P}_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \leq n\}$. In this paper, we derive some interesting identities of higher-order Bernoulli and Euler polynomials arising from the properties of those bases for $\mathbb{P}_n$.

1 Introduction

For $r \in \mathbb{R}$, let us define the Bernoulli polynomials of order r as follows:

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [1–18])}. \tag{1}$$

In the special case, $x = 0$, $B_n^{(r)}(0) = B_n^{(r)}$ are called the nth Bernoulli numbers of order r. As is well known, the Euler polynomials of order r are defined by the generating function to be

$$\left(\frac{2}{e^t + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!} \quad \text{(see [1–10])}. \tag{2}$$

For $\lambda\ (\neq 1) \in \mathbb{C}$, the Frobenius-Euler polynomials of order r are also given by

$$\left(\frac{1-\lambda}{e^t - \lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda)\frac{t^n}{n!} \quad \text{(see [1, 7])}. \tag{3}$$

The Hermite polynomials are defined by the generating function to be:

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \quad \text{(see [8–10, 19])}. \tag{4}$$

Thus, by (4), we get

$$H_n(x) = (H + 2x)^n = \sum_{l=0}^{n} \binom{n}{l} H_{n-l} 2^l x^l \quad \text{(see [14])}, \tag{5}$$

where $H_n = H_n(0)$ are called the nth Hermite numbers. Let $\mathbb{P}_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \le n\}$. Then $\mathbb{P}_n$ is an $(n+1)$-dimensional vector space over $\mathbb{Q}$. In [8, 10], it is called that $\{E_0^{(r)}(x), E_1^{(r)}(x), \ldots, E_n^{(r)}(x)\}$ and $\{B_0^{(r)}(x), B_1^{(r)}(x), \ldots, B_n^{(r)}(x)\}$ are bases for $\mathbb{P}_n$. Let Ω denote the space of real-valued differential functions on $(\infty, -\infty) = \mathbb{R}$. We define four linear operators on Ω as follows:

$$I(f)(x) = \int_x^{x+1} f(x)\,dx, \qquad \Delta(f)(x) = f(x+1) - f(x), \tag{6}$$

$$\tilde{\Delta}(f)(x) = f(x+1) + f(x), \qquad D(f)(x) = f'(x). \tag{7}$$

Thus, by (6) and (7), we get

$$I^n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-l} f_n(x+l) \quad \text{(see [8, 10, 12, 13])}, \tag{8}$$

where $f_1' = f, f_2' = f_1, \ldots, f_n' = f_{n-1}$, $n \in \mathbb{N}$.

In this paper, we derive some new interesting identities of higher-order Bernoulli, Euler and Hermite polynomials arising from the properties of bases of higher-order Bernoulli and Euler polynomials for $\mathbb{P}_n$.

2 Some identities of higher-order Bernoulli and Euler polynomials

First, we introduce the following theorems, which are important in deriving our results in this paper.

Theorem 1 [8] *For $r \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, let $p(x) \in \mathbb{P}_n$. Then we have*

$$p(x) = \frac{1}{2^r} \sum_{k=0}^{n} \sum_{j=0}^{r} \frac{1}{k!} \binom{r}{j} D^k p(j) E_k^{(r)}(x).$$

Theorem 2 [10] *For $r \in \mathbb{Z}_+$, let $p(x) \in \mathbb{P}_n$:*

(a) *If $r > n$, then we have*

$$p(x) = \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{1}{k!} (-1)^{k-j} \binom{k}{j} \left(I^{r-k} p(j)\right) B_k^{(r)}(x).$$

(b) *If $r \le n$, then*

$$\begin{aligned} p(x) = {} & \sum_{k=0}^{r-1} \sum_{j=0}^{k} \frac{1}{k!} (-1)^{k-j} \binom{k}{j} \left(I^{r-k} p(j)\right) B_k^{(r)}(x) \\ & + \sum_{k=r}^{n} \sum_{j=0}^{r} \frac{1}{k!} (-1)^{r-j} \binom{k}{j} \left(D^{k-r} p(j)\right) B_k^{(r)}(x). \end{aligned}$$

Let us take $p(x) = H_n(x) \in \mathbb{P}_n$.

Then, by (5), we get

$$\begin{aligned} p^{(k)}(x) = D^k p(x) &= 2^k n(n-1)\cdots(n-k+1)H_{n-k}(x) \\ &= 2^k \frac{n!}{(n-k)!} H_{n-k}(x). \end{aligned} \tag{9}$$

From Theorem 1 and (9), we can derive the following equation (10):

$$\begin{aligned} H_n(x) &= \frac{1}{2^r}\sum_{k=0}^{n}\left\{\sum_{j=0}^{r}\frac{1}{k!}\binom{r}{j}2^k\frac{n!}{(n-k)!}H_{n-k}(j)\right\}E_k^{(r)}(x) \\ &= \frac{1}{2^r}\sum_{k=0}^{n}\binom{n}{k}2^k\left[\sum_{j=0}^{r}\binom{r}{j}H_{n-k}(j)\right]E_k^{(r)}(x). \end{aligned} \tag{10}$$

Therefore, by (10), we obtain the following theorem.

Theorem 3 *For $n, r \in \mathbb{Z}_+$, we have*

$$H_n(x) = \frac{1}{2^r}\sum_{k=0}^{n}\binom{n}{k}2^k\left[\sum_{j=0}^{r}\binom{r}{j}H_{n-k}(j)\right]E_k^{(r)}(x).$$

We recall an explicit expression for Hermite polynomials as follows:

$$H_n(x) = \sum_{l=0}^{[\frac{n}{2}]}\frac{(-1)^l n!}{l!(n-2l)!}(2x)^{n-2l}. \tag{11}$$

By (11), we get

$$H_{n-k}(j) = \sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-1)^l (n-k)!}{l!(n-k-2l)!}(2j)^{n-k-2l}. \tag{12}$$

Thus, by Theorem 3 and (12), we obtain the following corollary.

Corollary 4 *For $n, r \in \mathbb{Z}_+$, we have*

$$H_n(x) = \frac{1}{2^r}\sum_{k=0}^{n}\left\{\sum_{j=0}^{r}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{\binom{n}{k}\binom{r}{j}2^k(-1)^l(n-k)!(2j)^{n-k-2l}}{l!(n-k-2l)!}\right\}E_k^{(r)}(x).$$

Now, we consider the identities of Hermite polynomials arising from the property of the basis of higher-order Bernoulli polynomials in $\mathbb{P}_n$.

For $r > k$, by (6) and (8), we get

$$\begin{aligned} I^{r-k}H_n(x) &= \sum_{l=0}^{r-k}\binom{r-k}{l}(-1)^{r-k-l}\frac{H_{n+r-k}(x+l)}{2^{r-k}(n+1)\cdots(n+r-k)} \\ &= \sum_{l=0}^{r-k}\binom{r-k}{l}(-1)^{r-k-l}\frac{n!H_{n+r-k}(x+l)}{2^{r-k}(n+r-k)!}. \end{aligned} \tag{13}$$

Therefore, by Theorem 2 and (13), we obtain the following theorem.

Theorem 5 *For $n, r \in \mathbb{Z}_+$, with $r > n$, we have*

$$H_n(x) = n! \sum_{k=0}^{n} \left\{ \sum_{j=0}^{k} \sum_{l=0}^{r-k} \frac{\binom{r-k}{l}\binom{k}{j}(-1)^{r-j-l} H_{n+r-k}(j+l)}{2^{r-k} k!(n+r-k)!} \right\} B_k^{(r)}(x).$$

Let us assume that $r, k \in \mathbb{Z}_+$, with $r \le n$. Then, by (b) of Theorem 2, we get

$$\begin{aligned} H_n(x) &= n! \sum_{k=0}^{r-1} \left\{ \sum_{j=0}^{k} \sum_{l=0}^{r-k} \frac{\binom{r-k}{l}\binom{k}{j}(-1)^{r-j-l} H_{n+r-k}(j+l)}{2^{r-k} k!(n+r-k)!} \right\} B_k^{(r)}(x) \\ &\quad + n! \sum_{k=r}^{n} \left\{ \sum_{j=0}^{r} \frac{(-1)^{r-j}\binom{r}{j} 2^{k-r} H_{n+r-k}(j)}{k!(n+r-k)!} \right\} B_k^{(r)}(x). \end{aligned} \tag{14}$$

Therefore, by (14), we obtain the following theorem.

Theorem 6 *For $n, r \in \mathbb{Z}_+$, with $r \le n$, we have*

$$\begin{aligned} H_n(x) &= n! \sum_{k=0}^{r-1} \left\{ \sum_{j=0}^{k} \sum_{l=0}^{r-k} \frac{\binom{r-k}{l}\binom{k}{j}(-1)^{r-j-l} H_{n+r-k}(j+l)}{2^{r-k} k!(n+r-k)!} \right\} B_k^{(r)}(x) \\ &\quad + n! \sum_{k=r}^{n} \left\{ \sum_{j=0}^{r} \frac{(-1)^{r-j}\binom{r}{j} 2^{k-r}}{k!(n+r-k)!} H_{n+r-k}(j) \right\} B_k^{(r)}(x). \end{aligned}$$

Remark From (12), we note that

$$H_{n+r-k}(j+l) = \sum_{m=0}^{[\frac{n+r-k}{2}]} \frac{(-1)^m (n+r-k)!}{m!(n+r-k-2m)!} (2j+2l)^{n+r-k-2m} \tag{15}$$

and

$$H_{n+r-k}(j) = \sum_{m=0}^{[\frac{n+r-k}{2}]} \frac{(-1)^m (n+r-k)!}{m!(n+r-k-2m)!} (2j)^{n+r-k-2m}. \tag{16}$$

Theorem 7 [10] *For $n, r \in \mathbb{Z}_+$, with $r > n$ and $p(x) \in \mathbb{P}_n$, we have*

$$p(x) = \sum_{k=0}^{n} \left\{ \sum_{j=0}^{k} \sum_{l=0}^{n} \frac{(r-k)! S_2(l+r-k, r-k)}{(l+r-k)! k!} (-1)^{k-j} \binom{k}{j} p^{(l)}(j) \right\} B_k^{(r)}(x),$$

where $S_2(l, n)$ is the Stirling number of the second kind and $p^{(l)}(j) = D^l p(j)$.

Theorem 8 [10] *For $n, r \in \mathbb{Z}_+$, with $r \le n$ and $p(x) \in \mathbb{P}_n$, we have*

$$\begin{aligned} p(x) &= \sum_{k=0}^{r-1} \left\{ \sum_{j=0}^{k} \sum_{l=0}^{n} \frac{(r-k)! S_2(l+r-k, r-k)}{(l+r-k)! k!} (-1)^{k-j} \binom{k}{j} p^{(l)}(j) \right\} B_k^{(r)}(x) \\ &\quad + \sum_{k=r}^{n} \left\{ \sum_{j=0}^{r} \frac{(-1)^{r-j}}{k!} \binom{r}{j} p^{(k-r)}(j) \right\} B_k^{(r)}(x). \end{aligned}$$

Let us take $p(x) = H_n(x) \in \mathbb{P}_n$. Then, by Theorem 7 and Theorem 8, we obtain the following corollary.

Corollary 9 *For $n, r \in \mathbb{Z}_+$:*

(a) *For $r > n$, we have*

$$H_n(x) = n! \sum_{k=0}^{n} \left\{ \sum_{j=0}^{k} \sum_{l=0}^{n} \frac{(r-k)! S_2(l+r-k, r-k)}{(l+r-k)! k! (n-l)!} (-1)^{k-j} \binom{k}{j} 2^l H_{n-l}(j) \right\} B_k^{(r)}(x).$$

(b) *For $r \leq n$, we have*

$$\begin{aligned} H_n(x) &= n! \sum_{k=0}^{r-1} \left\{ \sum_{j=0}^{k} \sum_{l=0}^{n} \frac{(r-k)! S_2(l+r-k, r-k)}{(l+r-k)! k! (n-l)!} (-1)^{k-j} \binom{k}{j} 2^l H_{n-l}(j) \right\} B_k^{(r)}(x) \\ &\quad + n! \sum_{k=r}^{n} \left\{ \sum_{j=0}^{r} \frac{(-1)^{r-j} \binom{r}{j} 2^{k-r} H_{n-k+r}(j)}{k! (n-k+r)!} \right\} B_k^{(r)}(x). \end{aligned}$$

Theorem 10 [9] *For $p(x) \in \mathbb{P}_n$, we have*

$$p(x) = \frac{1}{(1-\lambda)^r} \sum_{k=0}^{n} \left\{ \sum_{j=0}^{r} \frac{1}{k!} \binom{r}{j} (-\lambda)^{r-j} p^{(k)}(j) \right\} H_k^{(r)}(x|\lambda).$$

Let us take $p(x) = H_n(x) \in \mathbb{P}_n$. Then

$$\begin{aligned} H_n(x) &= \frac{1}{(1-\lambda)^r} \sum_{k=0}^{n} \left\{ \sum_{j=0}^{r} \frac{1}{k!} \binom{r}{j} (-\lambda)^{r-j} 2^k \frac{n!}{(n-k)!} H_{n-k}(j) \right\} H_k^{(r)}(x|\lambda) \\ &= \frac{1}{(1-\lambda)^r} \sum_{k=0}^{n} \binom{n}{k} 2^k \left\{ \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} H_{n-k}(j) \right\} H_k^{(r)}(x|\lambda) \\ &= \frac{1}{(1-\lambda)^r} \sum_{k=0}^{n} \left\{ \sum_{j=0}^{r} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{\binom{n}{k} 2^k \binom{r}{j} (-\lambda)^{r-j} (-1)^l (n-k)! (2j)^{n-k-2l}}{l! (n-k-2l)!} \right\} H_k^{(r)}(x|\lambda). \end{aligned} \tag{17}$$

Therefore, by (17), we obtain the following corollary.

Corollary 11 *For $n \in \mathbb{Z}_+$, we have*

$$H_n(x) = \frac{1}{(1-\lambda)^r} \sum_{k=0}^{n} \left\{ \sum_{j=0}^{r} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{\binom{n}{k} 2^k \binom{r}{j} (-\lambda)^{r-j} (-1)^l (n-k)! (2j)^{n-k-2l}}{l! (n-k-2l)!} \right\} H_k^{(r)}(x|\lambda).$$

For $r = 1$, the Frobenius-Euler polynomials are defined by the generating function to be

$$\left(\frac{1-\lambda}{e^t - \lambda} \right) e^{xt} = \sum_{n=0}^{\infty} H_n(x|\lambda) \frac{t^n}{n!} \quad \text{(see [9]).} \tag{18}$$

Thus, by (18), we get

$$\frac{d}{d\lambda} H_n(x|\lambda) = \frac{1}{1-\lambda} \left(H_n^{(2)}(x|\lambda) - H_n(x|\lambda) \right). \tag{19}$$

For $n \in \mathbb{Z}_+$, let $p(x) \in \mathbb{P}_n$. Then we note that

$$(1-\lambda)p(x) = \sum_{k=0}^{n} \frac{1}{k!}\left\{p^{(k)}(1) - \lambda p^{(k)}(0)\right\}H_k(x|\lambda) \quad \text{(see [9])}. \tag{20}$$

Let us take $p(x) = H_n(x)$. Then, by (20), we get

$$\begin{aligned}(1-\lambda)H_n(x) &= \sum_{k=0}^{n} \frac{1}{k!}\left\{2^k \frac{n!}{(n-k)!}H_{n-k}(1) - \lambda 2^k \frac{n!}{(n-k)!}H_{n-k}\right\}H_k(x|\lambda) \\ &= \sum_{k=0}^{n} \binom{n}{k} 2^k \left(H_{n-k}(1) - \lambda H_{n-k}\right)H_k(x|\lambda) \quad \text{(see [9])}. \end{aligned}\tag{21}$$

Therefore, by (21), we obtain the following theorem.

Theorem 12 *For $n \in \mathbb{Z}_+$, we have*

$$(1-\lambda)H_n(x) = \sum_{k=0}^{n} \binom{n}{k} 2^k \left(H_{n-k}(1) - \lambda H_{n-k}\right)H_k(x|\lambda).$$

Let us take $\frac{d}{d\lambda}$ on the both sides of Theorem 12.
Then, we have

$$\begin{aligned}-H_n(x) &= -\sum_{k=0}^{n} \binom{n}{k} 2^k H_{n-k}H_k(x|\lambda) \\ &\quad + \sum_{k=0}^{n} \binom{n}{k} 2^k \left(H_{n-k}(1) - \lambda H_{n-k}\right)\left(\frac{d}{d\lambda}H_k(x|\lambda)\right). \end{aligned}\tag{22}$$

By (22), we get

$$\begin{aligned}H_n(x) &= \sum_{k=0}^{n} \binom{n}{k} 2^k H_{n-k}H_k(x|\lambda) \\ &\quad + \sum_{k=0}^{n} \binom{n}{k} \left(\lambda H_{n-k} - H_{n-k}(1)\right) 2^k \left(\frac{d}{d\lambda}H_k(x|\lambda)\right). \end{aligned}\tag{23}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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